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PENROSE TILING P3: A DEEP DIVE INTO CONSTRUCTION RULES AND MATLAB INTEGRATION

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Abstract: *Quasicrystals, a fascinating class of materials discovered in 1982 by Israeli physicist Daniel Shechtman, have since captivated the scientific community with their exceptional properties. These structures exhibit a quasi-periodic translational sequence and possess rotational symmetry axes of 5, 8, 10, or 12—features that were previously considered unattainable in both conventional crystals and non-crystalline substances. This paper delves into the prevalent structural models of quasicrystals, spanning one-dimensional, two-dimensional, and three-dimensional representations. The one-dimensional model prominently features the quasi-periodic Fibonacci sequence, which has undergone extensive development from both experimental and theoretical standpoints. Among the various theoretical models for two-dimensional quasicrystals, this work focuses primarily on the Penrose tiling within the mosaic model—a quintessential example representing fivefold symmetric quasicrystal structures. In the realm of quasicrystals, piecework models come to the fore, characterized by the tessellation of two or more pieced blocks. The construction rules governing these models dictate that no overlap or coverage should exist between the pieced blocks, thereby forming a two-dimensional quasi-periodic structure devoid of gaps. The Penrose tiling, an iconic representative of this model, exhibits long-range order akin to crystal lattices, yet it lacks translational symmetry. It remains a widely studied quasicrystal model renowned for its distinctive fivefold symmetry.*

Keywords: *Quasicrystals, Structural Models, Penrose Tiling, Fibonacci Sequence, Crystallography*

1. Introduction

Quasicrystals were discovered in 1982 and reported for the first time in 1984. Daniel Shechtman, an Israeli physicist, discovered it. The structures of quasicrystals have a quasi-periodic translational sequence and can have rotational symmetry axes of 5, 8, 10, or 12, which are impossible in crystals and non-crystals [1].

The predominant structural models of quasicrystals are one-dimensional, two-dimensional, and three-dimensional models. The one-dimensional model primarily refers to the quasi-periodic Fibonacci sequence, which is highly developed from an experimental and theoretical standpoint. Two primary theoretical models for two-dimensional quasicrystals are the mosaic model and the overlay model. However, this paper only discusses the Penrose tiling of the mosaic model, a typical mosaic model for five-times symmetric quasicrystal structures [2].

The piecework models are formed by the tessellation of two or more pieced blocks, with the construction rules requiring that there be no overlap or coverage between the pieced blocks to form a two-dimensional quasi-periodic structure with no gaps

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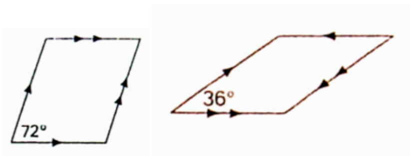
[3]. This model is represented by the Penrose tiling, which possesses the long-range orderliness of crystal punctures, lacks translational symmetry, and is a widely studied quasicrystal model with fivefold symmetry [4].

2. P3 Penrose tiling construction rules

Penrose tiling is representative of the two-dimensional quasicrystal tessellation model, which is a typical non-periodic tiling, and there are many different manifestations of this model. The most studied manifestation is the thick and thin rhombus model, also known as P3 Penrose tiling, and the other two models are P1 and P2 Penrose tiling, respectively. The P1 Penrose tiling consists of four types of pieces: the pentagon [5], the pentagram, the "boat," and the 36-degree skinny rhombus, whereas the P2 Penrose tiling consists of the "kite" and the "dart."(Figure 1).



(a) *(b)*

Figure 1: A thick rhombus (a) and a thin rhombus (b), which can be used to construct P3 Penrose tiling.

As shown in Figure 2-1, the P3 Penrose tiling consists of a thick rhombus with an acute angle of 72°and a thin rhombus with an acute angle of 36°. For these two types of rhombuses to form a P3 Penrose tiling, they must adhere to specific matching rules. Single and double arrows are drawn on the sides of the rhombuses, adjacent rhombuses have identical arrows on their common sides, and the number and direction of the rhombuses are identical. As shown in Figure 2, this matching rule can result in a P3 Penrose tiling that is both non-periodic and gap-free[6].

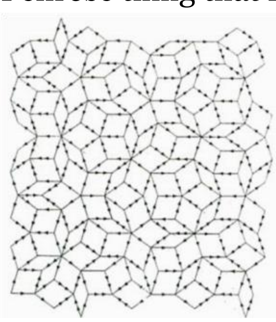


Figure 2: P3 Penrose tiling formed with thick and thin rhombuses, following the matching rule.

Matching, self-similar transformation, generalized pairwise, and high-dimensional projection methods are the most popular techniques for generating P3 Penrose tiling. This paper presents the theory and programming implementation of the self-similar transformation method used to generate P3 Penrose tiling [7].

3. Construction of P3 Penrose tiling by self-similar transformation method

The self-similar transformation method, whose generation mechanism is based on the self-similarity of Penrose tiling, generates a tiling that is self-similar to the tiling prior to the transformation after some alternative transformation. Specifically, beginning with a particular small tiling, after a substitution transformation, the resulting tiling is self-similar to the one before the transformation. Then the transformed tiling is enlarged to its original size. Repeating the two processes described above

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can generate a quasi-periodic tiling of arbitrary size. Multiple generations of this tiling can be generated by repeating the substitution method[8].

If each thick rhombus or thin rhombus in the Penrose tiling is If each thick or thin rhombus in the Penrose tiling is replaced according to the rules illustrated in Figure 3, the resulting graph is still a Penrose tiling[5]. The transformed tiling is self-similar to that before transformation, except that the side length

of the rhombus is shortened to 1/τ of the original length, where$\tau = \frac{\sqrt{5}+1}{2}$ (τ is the golden mean).

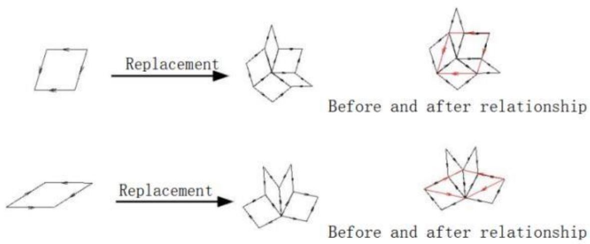

Figure 3: Construction rules for P3 Penrose tiling (starting with a thick rhombus and a thin rhombus, respectively).

Using this substitution rule, one can begin with a diamond-shaped tiling and replace it with a smaller diamond of the scale of the original puzzle$\frac{1}{\tau}$, then enlarge the new puzzle τ by a factor of two and repeat the process (shrink and expand) indefinitely to generate any generation of Penrose tiling. Thus, the method is also known as the shrink-expand method [9].

To generate the P3 Penrose tiling for previous generations, you must first develop a generation algorithm. The first step is drawing a thick rhombus (or a thin rhombus) and determining the center position and orientation of the first-generation rhombus structure based on the geometric relationship [10].

Then, based on the geometric relationship, we can investigate the relative position of the secondgeneration structure relative to the center of the first-generation rhombus and the relative bias angle of the second-generation structure relative to the orientation of the first-generation rhombus, following the self-similar transformation rule in the previous-generation descendant relationship, to draw the secondgeneration tiling[11].

To implement the expansion process for the second-generation structure, the edge length of the rhombus structure is enlarged τ times (τ is the golden ratio, approximately 1.618). Through the selfsimilar transformation rule, the center position and orientation of the third-generation structure are then determined based on the center position and orientation of each rhombus in the second-generation structure (the relative position of the third generation relative to the center of the second-generation rhombus, the third generation relative to the orientation of the second-generation rhombus, and so on)[12].

It is important to note that when drawing a figure, the rule of symmetry and the coordinates of the symmetry point can be considered. For instance, if you need to locate a point p3 in the tiling (assume p3 is a point of symmetry about the line connected by p1 and p2), you can first locate the symmetry point about the line connected by p1 and p2 to determine the center position, and then locate the coordinates of point p3[13]. Before executing the preceding procedure, we must define the expressions

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of two variables. Then, assuming that the two variables are set as xx and yy, we can use the formula for midpoint coordinates to determine the coordinate position of point p3[14].

If you need to determine the coordinates of the symmetry point (assuming the coordinates are [XX, YY]), you can begin by creating a splicing matrix (assuming the splicing matrix is [XC, YC]) and assigning values to [XC, YC]. The variables to be determined are X and Y[15].

Using the vertical foot coordinates, the results of XC2 and YC2 can be expressed as floating-point numbers. First, define the expressions of the variables XX and YY, then use the midpoint coordinate formula to determine the symmetry point's coordinates[16].

Figures 4 and Figures 5 depict the first seven generations of P3 Penrose tiling obtained by the authors of this study utilizing MATLAB software, beginning with a thin rhombus (a) and a thick rhombus (b), respectively, after six self-similar transformations[17].

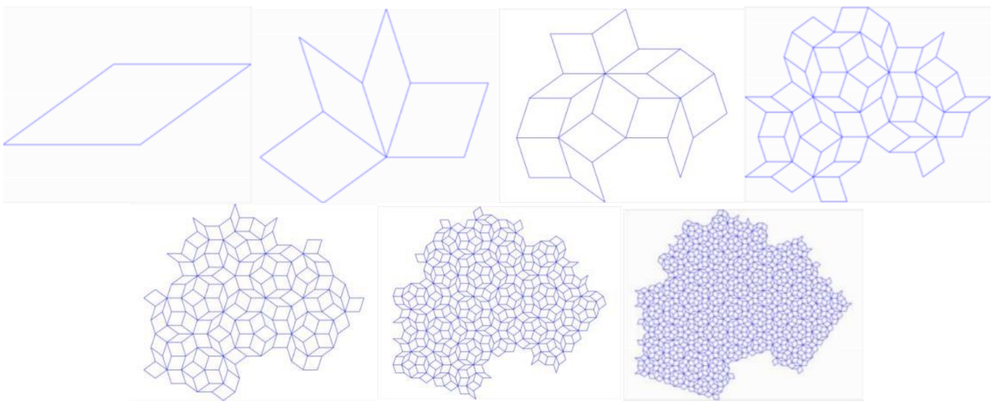

Figure 4: The first seven generations of the generated P3 Penrose tiling, starting with a thin rhombus (a).

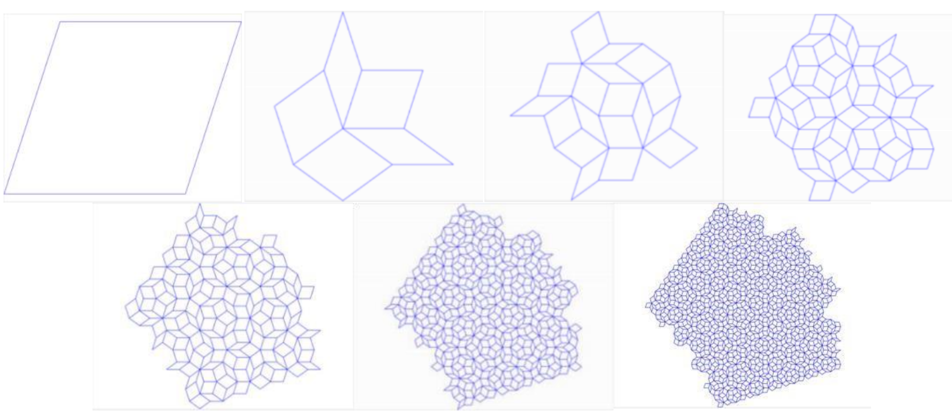

Figure 5: The first seven generations of the generated P3 Penrose tiling, starting with a thick rhombus (b).

According to theory, the self-similar transformation method can generate arbitrary generations of P3 Penrose tiling. The generation of arbitrary generation tiling cannot be fully realized in practice, however, due to the limited memory space of computers and the problem of objective programming errors. Moreover, if tiling of the eighth generation and beyond must be generated, the computer will generate more time and memory, which is challenging to implement [18].

4. Conclusion

This paper begins with the matching rules of P3 Penrose tiling. Then, it investigates the construction principle of the self-similar transformation method to generate Penrose tiling, designs the algorithm,

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and programs the construction of the first seven generations of the graph of P3 Penrose tiling using MATLAB software. The algorithm for generation can be broken down into the following steps: A thick or thin rhombus is initially drawn to determine the center position and orientation of the rhombus of the first generation. Secondly, the second-generation tiling is drawn by determining the relative position of the second-generation structure concerning the center of the first-generation rhombus and the relative deflection angle of the second-generation structure concerning the direction of the first-generation rhombus based on the geometric relationship. Third, to realize the expansion process, the edge length of the rhombus in the second generation structure is enlarged by τ times (τ is the golden ratio, approximately 1.618) Then[19], based on the center position and orientation of each rhombus in the second-generation structure, the self-similar transformation rule is used to determine the center position and orientation of the third-generation structure (the relative position of the third generation relative to the center of the second-generation rhombus and the relative deflection angle of the third generation relative to the orientation of the second-generation rhombus, and so on) to generate the P3 Penrose tiling for previous generations. If the algorithm designed by the authors of this paper is utilized to continue programming the computation, it is possible to generate the structure of the eighth generation and beyond. However, considering the computer's limited memory space and much time spent, it is challenging to implement the structure of the eighth generation and beyond in practice. Therefore, this paper only demonstrates the construction of the P3 Penrose tiling for the first seven generations. The P3 Penrose tiling is the model of tessellation that has been the subject of the most research. By examining the matching rules and constructing two-dimensional quasicrystals using the self-similar transformation method, we can gain a deeper understanding of the formation mechanism. Therefore, physical properties such as mechanics, thermodynamics, electricity, magnetism, and optics should be studied further.

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