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# EXTENDING THE SCOPE OF PIECEWISE LINEAR FUNCTION TECHNIQUES

***Dr. Lorenzo Matteo Bianchi***
Department Disag, University Of Siena, Piazza S.Francesco, 53100 Siena, Italy

***Abstract:*** *This paper explores a broader class of piecewise linear functions, extending their applicability beyond conventional domains. Piecewise linear functions are typically defined on closed convex domains, but this work introduces a more versatile set of maps known as SW(E m ,T). These maps are linear only on selected subsets of vectors and components, making them suitable for a wider range of applications.*
*The paper establishes an exponential function, F, which maps linear spaces to the set SW(E m ,T). It rigorously proves the uniqueness and existence of a universal element, denoted as *, within this framework. Furthermore, the paper introduces r-subset wise linear skew symmetric maps denoted as Φ = ∑ λμνϕ, demonstrating that they can be fully characterized by their values for λμν and a basis of E.*
*The concept of an r-determinant function is introduced, defined as an r-subset wise linear skew symmetric map Φ: E m → Γ, with Γ being an arbitrary field of characteristic 0. The paper delves into various properties of r-determinant maps, shedding light on their characteristics and utility.*
*Additionally, the paper explores the adjoint of a linear map ψ ∈ L(E,F), where E and F represent linear spaces. It also discusses the development of an r-determinant function using r-cofactors.*
*Furthermore, this work defines extensions of differential forms through r-subset wise skew symmetric maps, paving the way for generalized differential forms. The paper investigates the basis and spaces of these generalized differential forms.*
***Keywords:*** *piecewise linear functions, SW(E m ,T), exponential function, r-subset wise linear skew symmetric maps, r-determinant function, adjoint, differential forms, generalized differential forms.*

## Introduction

Piecewise linear functions are useful in several different contexts, piecewise linear manifolds, computer science or convex analysis are examples. A definition of a piecewise linear function is the following, see [8]. Let *C* a closed convex domain in□$^d$ , a function □:*C* □□ is said to be piecewise linear if there is a finite family *Q* of closed domains such that *C* = □*Q* and □ is linear on every domain in *Q* . A linear function □ on □$^d$ which coincides with □ on some $Q_i$ □*Q* is said to be a component of □. In this paper is considered a more general class of piecewise linear functions. It is defined the set of maps *SW*(*E* $^m$ ,*T*) which are linear only on a subset of $^r$ vectors and components.
Then an exponential function *F* is defined from linear spaces to the set *SW*(*E* $^m$ ,*T*). It is proved the uniqueness and existence of a function * as universal element for the function *F* . It is defined a r-subset wise linear skew symmetric □ = □□,□ □□□□ map and it is proved that this is completely determined by its values for □□□ and on a basis of *E* . A r-determinant function is defined as a r-subset wise linear skew symmetric map □ : *E* $^m$ □□ , where □ is an arbitrary field of characteristic 0. Some properties of r-determinant maps are considered. It is defined the adjoint for a linear map □□*L*(*E*,*F*), where *E* and *F* are linear spaces, and the development

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of a r-determinant function by r- cofactors. Extensions of differential forms are defined by r-subset wise skew symmetric maps. Basis and spaces of generalized differential forms are studied.

## 2. R-Subset wise Linear Mappings

Some properties of linear functions are extended to mappings which are linear only on subsets of r variables. □ Denotes an arbitrarily chosen field such that*char*□ = 0 .

The multindex $I_r^n$ of lenght$^r$ is defined by

$I_r^n$ ={($i_1$,□,$i_r$): 1□ $i_1$ < $i_2$ <□< $i_r$ □ $n$}

Besides, for a fixed natural $k$

($I_r^n$)$_k$ = {($i_1$,□,$i_p$ ,□,$i_r$ ):1□ $i_1$ <□< $i_p$ = $k$ <□□ $i_r$ □ $n$, where1□ $k$ □ $n$}

for the indices $j_1$,□, $j_k$ □$I_k^n$

($I_r^n$) $_{j,□,j}$ = {($i_1$,□,$i_p$ ,□,$i_p$ ,□,$i_r$ ) :

1 $k$ 1 $k$

1□ $i_1$ <□< $i_p$ = $j_1$ <□< $i_p$ = $j_k$ <□□ $i_r$ □ $n$}

1 $k$

Let {$^e$□} be a basis of an n-dimensional vector space $E$ and let $x$□ = □□$^n$=1$^x$□□$^e$□ be vectors of $E$ , $n$ □1.

**Definition 2.1***Let L(E $^r$ ,T) be the space of linear mappings of E$^r$ into the vector space T . Consider a mapping*

□□ : $E^m$ □$T$

□

□□ : ($x_1$,□,$x_m$ ) □□□□□□($x$□□$^1$$e$□,□,$x$□□$^r$ $e$□) 1□ $r$ □ $n$,1□ $r$ □ $m$, □□□ □□

□□ □,□

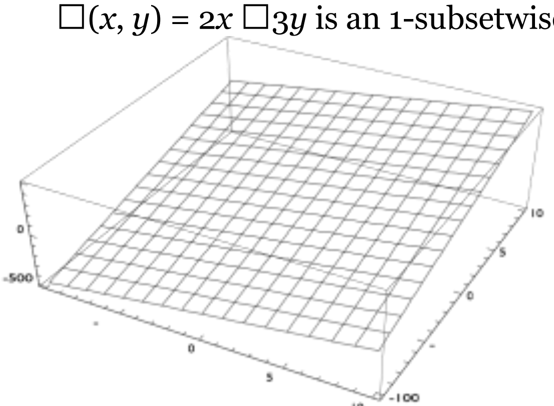
Where the sum is over every system of indices □= □$_1$,□,□$_r$ □$I_r^m$ , □ =□$_1$,□,□$_r$ □$I_r^n$ . If $n$=$m$ then $^{r\,<}$ $_{n\,=\,m}$ . The sum ($x$□□$^i$ $e$□ □□□ $x$□□$^i$ $e$□ ) is denoted in short by $x$□□$^i$ $e$□, and □: $E^r$ □$T$ is an r-linear mapping.

1 1 $r$ $r$

Then □ is said to be r-linear with respect to the r-subsets of vectors and components, that is, an r-subsetwise linear mapping. The linear mappings □ are the components of □.

**Example 2.1** *The function* □:□$^1$□$^2$ □□ *defined by*

□($x$, $y$) = 2$x$ □3$y$ is an 1-subsetwise linear function.




Graph of the function Φ. (Obtained by Mathematica).

**Example 2.2** *The map* □ :(□$^2$ )$^3$ □ □$^2$□$^2$ *defined by*

□$x$ $x12$ □$x13$ □$x_{13}$□
□[($x11$, $x21$),($x12$, $x22$ ),($x13$, $13$□$x1$ $23$□$x1$ □□ $x_{23}$□
$x23$)]=□12□□$x1121$ 1 2 □□□□ □□□□ □□□□
□ □□□□ □□□□
□□$x21$ □□$x22$
$x_{22}$□ □ $x_{23}$□ □

is an 2-subsetwise linear map.

**Example 2.3** *Let $f_1$,□, $f_r$ be a linearly independent set of the space $L(E^r,T)$ , a r-subsetwise linear map is defined by*

□($x1$,□,$xm$ ) = □□□□( $f1$($x$□□1$e$□)□ $f2$ ($x$□□2 $e$□)□□$fr$□□□□□
($x$□□$r$ $e$□))
□,□

**Theorem 2.1** *An r-subsetwise linear mapping □, with $r < m$, is not linear Proof.* For any r-subsetwise linear mapping □, $r < m$,
□($x1$,□,$xi$ □$yi$,□,$xm$)=□□□□□($x$□□1$e$□,□,$x$□$i$ $e$□,□,$x$□□ $r$ $e$□)
□□□□□□($x$□□1$e$□,□,$y$□$i$ $e$□,□,$x$□□$r$ $e$□)
□,□ □,□
□ □($x_1$,□,$x_i$ ,□,$x_m$ )□□($x_1$,□, $y_i$ ,□, $x_m$ )

In the first sum on the right side □= □$_1$,□,$i$,□,□$_r$ □$I_r^m$ . Unlike, in the second sum □= □$_1$,□,$i$,□,□$_r$ □$(I_r^m)_i$ , so this sum cannot be □($x_1$,□, $y_i$ ,□,$x_m$ ) . □

As a special case, if $r$=$m$ then □ is linear.

If $t:T$ □ $H$ is linear and □ is r-swlin (subsetwise linear) map, then

$t$□ = $t$(□□$_□$$^□$□) = □□$_□$$^□$$t$□

and $t$□ is a r-swlin map.

By the set $SW(E^m,T)$ of the r-swlin maps, the following exponential functor $F$ , from linear spaces to sets, is defined by

$F(T) = SW(E^m,T)$ for any linear space T
□$F(t)$: $F(T)$ □ $F(H)$
□ for any linear t :T □ H
□$F(t)$:$\Phi$□□$t$□$^\Phi$

**Theorem 2.2** *For any r-swlin mapping □ : $E^m$□ $H$ there exists a unique linear mapping $f$ : $E$□□□$E$ □ $H$ such that*

$f(x_1$□□□ $x_m) =$ □$(x_1$,□,$x_m)$

That is, the mapping □: $E^m$□$T$ is an universal element for the functor $F$ .

*Proof.* The proof generalizes to swlin maps the classical proof of universality of the tensor product, see [4], [6].

Uniqueness. Suppose that □: $E^m$□$T$ and ~□ : $E^m$□$T$~ are universal elements for the functor $F$ , then, there exist linear maps

$\tilde{f}:T$□$\tilde{T}$ and $\tilde{g}:T$□$T$

suchthat

$f$ ($x1$ □□□$xm$ ) = $x1$ and ~ $g$($x1$ □□~□ $xm$ ) = $x1$
~□□~□ $xm$ □□□$x m$

thatis

$gf$ ($x_1$□□□ $x_m$ ) = $x_1$□□□ and ~ $fg$($x1$ □□~□ $xm$ ) = $x1$
$x_m$ ~□□~ □ $x m$

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~
by the universality of □ and □ it follows, respectively

$1T = g$ □□$f$ and $1\,T\sim = f$ □□$g$

thus $f$ and $g$ are inverse linear isomorphisms.

**Existence:** Consider the free vector space $C(E^r)$ generated by the space $E^r$. Denote by $N(E^r)$ the subspace of $C(E^r)$ spanned by the vectors

($x$□□1$e$□,□,□1 $y1$ □□2 $y2$,□, $x$□□$r$ $e$□) □□1($x$□□ 1$e$□,□, $y1$,□, $x$□□ $r$ $e$□)
□□2 ($x$□□ 1$e$□,□$y2$ ,□, $x$□□ $r$ $e$□)

for □= □₁,□,□$_r$ □$I_r^m$ , □ =□₁,□,□$_r$ □$I_r^n$ , □$_i$ □□ and $x$□□$^r$ $e$□, $y_1$, $y_2$ □$E^r$ .

Set $S = C(E^r)/N(E^r)$ and let □:$C(E^r)$ □ $S$ be the canonical projection. Define the map

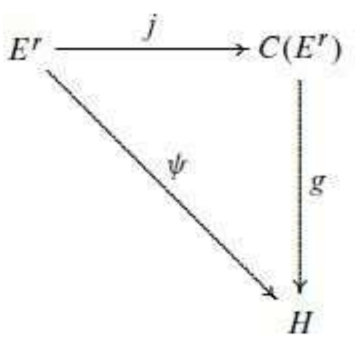
□□: $E^m$ □ $S$
□ ($x1$,□,$xm$ )
□□: □□□□□□($x$□□1$e$□,□,$x$□□$r$
□□ $e$□ )
□,□

Since □ is a homomorphism, it follows that □ is an r-swlin map.
If $z$□$S$ , then it is a finite sum

$z$ = □□□(□□□□□($x$□□1$e$□,□,$x$□□ $r$ $e$□))□
□ □,□
= □□□ ($x_1$ □□□$x_m$ ) □
□

so □$z$□$S$ , $^z$ is spanned by the products $x_1$ □□□ $x_m$ and $I_m$□ = $S$ .

Moreover let □:E$^r$ □ $H$ be a r-linear map. Since $C(E^r)$ is a free vector space, there exists an unique linear map $g$ such that the following diagram commutes




where $j$ is the insertion of $E^r$ in $C(E^r)$ . So

$g$($x$□□ 1$e$□,□,$x$□□$r$ $e$□) =□($x$□□1$e$□,□,$x$□□ $r$ $e$□)

If

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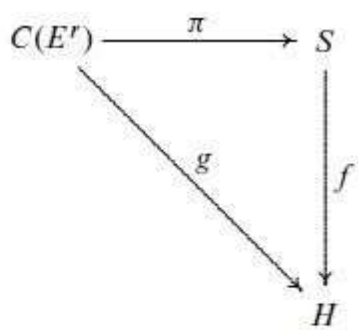
$z$ = ($x$□□ 1$e$□,□,□1 $y$1 □□2 $y$2 ,□, $x$□□$r$ $e$□)□□1($x$□□1$e$□,□, $y$1,□,$x$□□ $r$ $e$□) □□2 ($x$□□1$e$□,□$y$2,□,$x$□□$r$ $e$□)

Is a generator of $N(E^r)$ , then

$g(z)$ =□($z$) =□($x$□□ 1$e$□,□,□1$y$1 □□2 $y$2,□,$x$□□$re$□)□□1□($x$□□1$e$□,□, $y$1,□,$x$□□ $re$□) □□2□($x$□□1$e$□,□$y$2,□,$x$□□$r$ $e$□)
= 0

then $N(E^r)$ □ $Kerg$ . For the principal theorem on factor spaces, see [5], there exists an unique linear map
$f$ such that the following diagram commutes




that is, □ is an universal element. So

($f$ □□)($x_1$,□,$x_m$ ) = $f$ (□□□□□($x$□□1$e$□,□, $x$□□ $r$ $e$□))
□,□
= □□□□$f$ □□($x$□□1$e$□,□,$x$□□ $r$ $e$□)
□,□
= □□□□$g$($x$□□1$e$□,□,$x$□□ $r$ $e$□)
□,□
= □□□□□($x$□□1$e$□,□,$x$□□ $r$ $e$□)
□,□
= □($x$1,□,$xm$ )

**Example 2.4** *Consider the 2-swlin function □ defined by* □

□□ : (□$^2$)$^3$ □□ □12
□ 12 13 23 ,□13,□23
□□ : ($x_1$, $x_2$ , $x_3$ ) □□ ($x_1$, $x_2$ ) □□ ($x_1$, $x_3$) □□ ($x_2$, $x_3$) □□

where the bilinear function (□,□), on the right side, is the inner product in □$^2$ . By the theorem 2.2, the map □:(□$^2$ )$^3$ □ □$^2$ □□$^2$ □□$^2$ is universal, so an unique linear function $f$ :□$^2$ □□$^2$ □□$^2$ □ □ exists such that $f$ ($x_1$ □$x_2$ □$x_3$) = □($x_1$,$x_2$ ,$x_3$ ). Since □$^2$ □□$^2$ □□$^2$ is free, the function $f$ is determined by its values $f$ ($x_1$ □ $x_2$ □ $x_3$) on the free generators $x_1$ □$x_2$ □$x_3$ .

**Corollary 2.1** *For any r-swlin map □ : $E^m$ □T*

□($x1$,□,$xm$ ) = □□□□($x$□□1e□ □□□ $x$□□ r e□)
□,□

□ □ □ □

*Proof*. Since □($x_{□}{}^{1}e_{□}$,□,$x_{□}{}^{r}e_{□}$) = $x_{□}{}^{1}e_{□}$ □□□ $x_{□}{}^{r}e_{□}$ , by the theorem 2.2

□($x_1$,□,$x_m$ ) = ( f □□)(($x_1$,□,$x_m$ ) = f (□□□□($x$□□1e□ □□□ $x$□□ r e□)
□,□

**Example 2.5** *Let □ :(□$^n$ )$^n$ □T be a 2-swlin map. The tensor product □:□$^n$ □□$^n$ □ $M^{n□n}$ is defined by $x_i$ □ $x_i$ = $x_i x'_i$ , $x_i$ □□$^n$ , see [4], then □:(□$^n$ )$^n$ □ □$^n$ □□□□$^n$ is given by*
1 2 1 2

$x1$ □□□$xn$ = □ □($i1$,$i2$ ) $xi1$ □ $x$ $i2$
($i$ ,$i$ )□$I^n$
1 2 2
□ □ □($i1$,$i2$) $x1i$ $x1i$ □□□ □($i1$,$i2$ )$x1i$ $xni$ □ □
□ 1 2 1 2
□ ($i_1$,$i_2$ )□$I_2{}^n$ ($i_1$,$i_2$)□$I_2{}^n$ □
=□ □ □ □ □
□□□□($i_1$,□$i_2$) $I$ $n$□($i1$,$i2$) $xni1$ $x1i2$ □□($i1$,□$i2$)□$I2n$□($i1$,$i2$) $xni1$ $xni2$ □□ □ □
□ 2

## 3.{r,□}- determinant

If □ is a permutation, □□$S_r$ , then the mapping □□:□$^r$ □ F is defined by
□□($x_1$,□,$x_r$ ) =□($x$□ ,□,$x$□ ) . More generally
1 r

**Definition 3.1** *Let □($x_1$,□,$x_m$ ) be an r-swlin map, for any permutation □□$S_r$ , the mapping □□ : $E^m$ □T , is defined by*
□□($x1$,□,$xm$ ) = □□□□□□($x$□□1e□,□,$x$□□r e□) = □□□□□□($x$□□(□1)e□,□,$x$□□(□r ) e□)
□,□ □,□

**Definition 3.2** *An r-swlin map □($x_1$,□,$x_m$ ) is said skewsymmetric if for any □□$S_r$ is □□ =□□□ where*
□□ =1(□□ = □1) *for any even (odd) permutation □.*

**Theorem 3.1** *An r-swlin map □ = □□□□□ is skewsymmetric if and only if □ is skewsymmetric.*
*Proof.*Suppose □ skewsymmetric, then
□□ = □□□□□□□($x$□□1e□,□,$x$□□r e□) = □□□□□□□□($x$□□1e□,□,$x$□□ r e□) =□□□
□,□ □,□
Conversely, □□ =□□□ implies

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□□□□□□= □□□□□□□
□,□ □,□
so □□, □ □□□(□□□□□□) = 0 for all $x$□ □1e□,□,$x$□□r e□, then □□=□□□. □

**Theorem 3.2** *Every r-swlin map* □($x_1$,□,$x_m$) *determines an r-swlinskewsymmetric map* □ *, given by*
□ = □□□□□ = □□□□□□□□□(x□□1e□,□, x□□ r e□)
□ □,□ □
where the second sum on right side is over all permutations □□$S_r$.
*Proof.* For any□□$S_r$
□□ = □□(□□□□□□□□) = □□□(□□□□□□□□) = □□(□□□□□□□□□) = □□□.
□,□ □ □,□ □ □,□ □
□

**Theorem 3.3** *Let* □ = □□, □ □□□□: *E* $^m$ □ *F be an r-swlinskewsymmetric map, then* □ *is completely determined by its values on a basis of E and by the constants* □□□.

*i* *n* *i* *i*
*Proof.* Let {e□} be a basis of *E* . Let *x* = □□=1x□e□, *i* =1,□,*m* be vectors in *E* and *X* = (x□) , then
*n* *n*
□($x_1$,□, $x_m$) = □(□x□¹e□,□,□x□ᵐe□)
□=1 □=1
*n* *n*
= □□□□□((□x□□1e□)□,□,(□x□□r e□)□) □□*Irn*, □□*Irm*
□,□ □=1 □=1
= □□□□( □ □□x□□1 □x□□ r □(e□ ,□,e□ )) □□*Sr*
□
□,□ □=□ ,□,□ 1 □r □1 □1
1 r
= □□□□ | X□□ |□(e□ ,□,e□ )
1 r
□,□
where X□□ is the square submatrix of *X* determined by rows indexed by □ and columns indexed by □.

**Example 3.1** *Let* □ :(□3)3 □ □3 *be a 2-swlin skewsymmetric map defined by*

□ x x □
□ ( x , x , x ) = □ j1 , j2□□ i1 , j1 i1 , j2 □
1 2 3 ( i1 ,i2 ), □1 2 2 i1 ,i2 □ x i , j x i , j □ □
( j , j )□ I 3 □ 2 1 2 2

3 3 where $x_i$ = □k=1$x_{k,i}e_k$ □□ . Then

j , j
□(x1, x2 , x3 ) = □ □i11,i22□(xi1 j1ei1 □ xi2 j1ei2 , xi1 j2 ei1 □ xi2 j2 ei2 )
(i1,i2 ),( j1, j2 )□I2³

□ j , j2□xi1, j1 x i1, j2 □(ei ,ei )

= □i1,i

1 2 x x 1 2
(i ,i ),( j , j )□I3 i2, j1 i2, j2
1 2 1 2 2

**Definition 3.3** *Let {e□} be a basis of E , then an r-swlinskewsymmetric map □E (x1,□, xm ) :E m □□ such that □(e□ ,□,e□ ) = 1, □□I r n , is said an r-determinant function.*
1 r

The scalar detr,□X = □□, □□□□ | X□□ | will be said the (r,□) -determinant of X = (x□i ) , relative to the basis
{e□}. If □□□ =| X□□ | we denote detr X =| X |r = □□,□ | X□□ |2 , see [2].

**Example 3.2** *In order to obtain a non-trivial example of r-determinant function, consider a 2-swlin function □ = □□,□ □□□□ defined by*
□(x1,□,xm ) = □□□□□e□□1 ,x□□1e□□□□e□□r ,x□□ r e□□
□,□ thatis
□(x□□ 1e□,□,x□□r e□) = □e□□1 ,x□□1e□□□□e□□r ,x□□ r e□□
where {e□},{e□□} are a pair of dual bases in E and E□ = L(E) ={f :f : E □□,f linear} respectively, with dimE = dimE□ □ r . The bilinear function □,□ is non-degenerate and it is defined by

□□ □ □□ □
□e i ,x□i e□□ = e i (x□i e□)

then □(x1,□,xm ) = □□□□□e□□1 ,x□□1e□ □□□e□□r ,x□□ r e□ □
1 1 r r
□

□
= □□□□x□□11 □x□□ r r
□

The set of the r-swlin maps is denoted by SW(E m ,T). The exponential functor F , from linear spaces to sets, is defined by

F(T) = SW(E m ,T) for any linear space T

□F(t): F(T) □ F(H)
□ for any linear t :T □ H
□F(t): □ □□t□

The following proposition states the universality of the r-determinant function.

**Theorem 3.4** *Let □E = □□,□□□□□: E m □□ be an r-determinant function in E , then for any r-swlinskewsymmetric mapping □ = □□□ □□□□: E m □ F , there is an unique vector f □F such that*
□(x1,□, xm ) = (□E (x1,□, xm )( f ) = □□□□ | X□□ | f□ □□I r m , □□Ir n , xi □E
□,□
where f□ are the components of the vector
f = (□(e 1 ,□,e□1r ),□,□(e □ □□□ nr □□□□,□,e□□□□□ nr □ □□ □ ))
□1
□1 r
□n□
i n and □ are the □□□ r□□□ elements of I r .

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*Proof*. Let $\{e_i\}$, $i = 1,\square,n$ be a basis of $E$ such that

$\square_E(x_1,\square, x_m) = \square\square_{\square}^{\square} | X_{\square}^{\square} |\square(e_{\square},\square,e_{\square}) = \square\square_{\square}^{\square} | X_{\square}^{\square} |$

1 r

□,□ □,□

that is , $\square(e_{\square},\square,e_{\square}) = 1$.

1 r

Then, for any r-swlin skew symmetric map

$\square(x_1,\square, x_m) = \square\square_{\square}^{\square} | X_{\square}^{\square} |\square = (\square_E(x_1,\square, x_m))(f)$

□,□ itfollows

$\square(e_{\square},\square,e_{\square}) = \square(e_{\square},\square,e_{\square})\square(e_{\square},\square,e_{\square}) = 1\square\square(e_{\square},\square,e_{\square})$

1 r 1 r 1 r 1 r

so □ and □ have the same values on the basis $\{e_{\square}\}$ and by theorem 3.3 it follows □ = □ . □

If $\square_E$ and $\square'_E$ are two r-determinant functions in $E$, then $\square\square_E\square\square\square'_E$, □,□□□ , is a r-determinant function too.

Let $\square_F$ be an r-determinant function in $F$ and let $\square: E \square F$ be a linear mapping of vector spaces, where $dimE = n$, $dimF = t$, then $\square_{\square} : E^m \square\square$, defined by

$\square_{\square}(x_1,\square,x_m) = \square_F(\square x_1,\square,\square x_m) = \square\square_{\square}^{\square}\square_F((\square x^{\square 1})_{\square},\square,(\square x^{\square r})_{\square})$

□,□

is an r-determinant function in $E$, where $\square_F : F^r \square\square$ is an r-linear mapping on $F$, $\square\square I_r^m$, $\square\square I_r^t$.

By theorem 3.4, $\square_{\square} = \square_F(f) = \square_{\square,\square,\square}\square\square_{\square}^{\square} | X_{\square}^{\square} | f_{\square}$ for an unique vector $f = (f_{\square})$.

Let $\square'_F$ be another nonnullswilin skew symmetric map, then

$\square'F = \square F(g) = \square\square\square\square | X\square\square | g\square$

□,□,□

and

$\square'_{\square} = \square_{\square}(g) = (\square_F(f))(g) = \square\square_{\square}^{\square} | X_{\square}^{\square} | f_{\square}g_{\square} = \square'_F(f_{\square})$

□,□,□

so the vector $f$ does not depend on the choise of $\square_F$ and it is determined by the map □, then the notation $f = det\square$ .

**Example 3.3** *Let □ and $A_{\square}$ be a linear map and its matrix respectively, defined by*

□1 0□

□□:□2 □□ 3 □ □

□ $A_{\square} = \square 0 \quad 1\square$

□□:$(x, y)$ □□$(x, y,x\square y)$ □□ 1 1□□

besides let $\square_3 : (\square^3)^3 \square\square$ be a 2-determinant function and $x_i \square\square^2$, then

□

$\square_{\square} = \square\square_3(\square x_1,\square x_2,\square x_3) = \square^{12}\square(\square x_1,\square x_2)\square\square^{13}\square(\square x_1,\square x_3)\square\square^{23}\square(\square x_2,\square x_3)$

2 2 2 2 2 2

$= \square 12\square(\square xi1\square ei,\square xi2\square ei)\square\square 13\square(\square xi1\square ei,\square xi3\square ei)\square\square 23\square(\square xi2\square ei,\square xi3\square ei)$

i=1 i=1 i=1 i=1 i=1 i=1

$= \square^{12} | X^{12} |\square(\square e_1,\square e_2)\square\square^{13} | X^{13} |\square(\square e_1,\square e_2)\square\square^{23} | X^{23} |\square(\square e_1,\square e_2)$

x x

ij1i 1 j

where| X |=. Since

x2i x 2j

1 0 1 0 0 1
□(□$e_1$,□$e_2$) =□((1,0,1),(0,1,1)) = □$_{12}$ | □□13 | □□23 |
0 1 1 1 1 1 =□12 □□13 □□23
then
□□ = □12 | $X$ 12 | $det2$,□□□□13 | $X$ 13 | $det2$,□□□□23 | $X$ 23 | $det2$,□□ = □□3 ($det2$,□□)
The expression for $det$□ may be obtained immediately by the matrix $A$□, see [2]
□1 0□
□ □1 01 00 1
$det2$,□$A$□ = $det2$,□□0 1□ | □□13 | □□23 | = □12=□12 □□13 □□23
□□ 1 1□□ 0 11 11 1

**Theorem 3.5** *Let □: E □ F be a linear mapping and $A$□ = (□□□) its matrix relative to the bases {$e$□},{$f$□},*
*□ =1,□,n , □ =1,□,t . Let □$_F$ = □□,□□□□□$_F$: $F^m$ □□ be an r-determinant function. If □$_F$ ( $f$□□1 ,□, $f$□□r ) =1, then*
i)
□□($x$1,□, $xm$ ) = □□□□(□ | $X$□□ || $A$□□ |) □□$I$ $rm$ ,□□$I$ $rn$ ,□□$I$ $r$ $t$
□,□ □
ii)
□□($e_1$,□,$e_n$ ) = □□□□ | $A$□□ |
□,□
where $A$□□ is the submatrix of $A$ determined by rows indexed by □ and columns indexed by □, for □ =□1,□,□$_r$ □$I$ $_r^n$ ,□ =□1,□,□$_r$ □$I$ $_r^t$ . The vectors $x_1$,□,$x_m$ , relative to the basis {$e$□ }, are expressed by $x$□ = □□$^n$ =1 $x$□□$e$□,□=1,□,$m$ and $X$ = ($x$□□) .

*Proof.* i)
n n
□□($x_1$,□,$x_m$ ) = □$_F$ (□$x_1$,□,□$x_m$ ) = □$_F$ (□$x$□$^1$□$e$□,□,□$x$□$^m$□$e$□)
□=1 □=1
n t n t
= □ $F$ (□$x$□1 □□1□ $f$□,□,□$x$□$m$ □□□$m$ $f$□)
□=1 □=1 □=1 □=1 t n t n
= □ $F$ (□(□$x$□1□□□) $f$□,□,□(□$x$□$m$□□□) $f$□)
□=1 □=1 □=1 □=1 n n
= □□□□□□$F$ (((□$x$□□1□□□) $f$□),□,((□$x$□□$r$□□□) $f$□) □□$I$ $rt$ , □□$Irm$
□,□ □=1 □=1
n □ n □
= □□□□( □ □□(□$x$□□1□□□1 )□(□$x$□□$r$□□□$r$ ))□$F$ ( $f$□□1 ,□, $f$□□ $r$ )
□,□ □=□1,□,□$r$ □=1 □=1
□□$S_r$ , by
n □ n □
□ □□(□$x$□□ 1□□□1 )□(□$x$□□$r$□□□$r$ ) = □ | $X$□□ || $A$□□ | it follows i).
□=□ ,□,□ □=1 □=1 □ 1 r

ii) It is a special case of i) for $X = I_n$ .

The scalar $det_{r,}$□□ = □□, □□□□ | $A$□□ | will be called the ($r$,□) -determinant of □, relative to the bases
{$e$□},{ $f$□}. If □□□ =| $A$□□ | , then □□,□ | $A$□□ |$^2$ will be denoted by $det_r$□ or |□ |$_r$
□

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**Theorem 3.6** *Let □:E □ F and □:F □ G be linear mappings of vector spaces. Let $□_F$ be a determinant function in*
*F . If $x_1$,□,$x_m$ are vectors in E , then*
□□□□($x1$,□, $xm$ ) = □□□□□($x1$,□, $xm$ )
*Proof.*
□□□□($x_1$,□, $x_m$ ) = $□_G$ (□□□($x_1$,□, $x_m$ )) = □□(□($x_1$,□,□$x_m$ )) = □□□□□($x_1$,□, $x_m$ )

**4. The (t,k)-forms**

Let $□^n_p$ be the tangent space of $□^n$ at the point $p$ and let $(□^n_p)^□$ be the dual space. Let $□^k(□^n_p)^□$ be the linear space of the k-linear alternating maps $□:(□^n_p)^k$ □□ , then denote by $□^k_t(□^n_p)^□$, with $k$ □ $t$ □ $n$ , the set of all k-linear alternating maps □: $(□^n_p)^t$ □□. The set $□^k_t(□^n_p)^□$ , by the usual operations of functions, is a linear space. If $□_1$,□,$□_t$ belong to $(□^n_p)^□$, then an element $□_1$ □□□$□_t$ □$□^k_t$ $(□^n_p)^□$ is obtained by setting

□1(v1)
($□_1$ □□□$□_t$)($v_1$,□,$v_k$ ) = $det_{k,□}□_i(v_j)$ = □ □1(vk
□(v ) t 1 □ )
□ □
$□_t(v_k)$

where $i$ =1,□,$t$ , $j$ =1,□,$k$ and $v_j$ □$□^n$ .

Observe that $□_1$ □□□$□_t$ is k-linear and alternate.

**Example 4.1** *When $□_1$,$□_2$ ,$□_3$ belong to $(□^3_p)^□$, an element $□_1$ □$□_2$ □$□_3$ □$□^2_3(□^3_p)^□$ is obtained by the 2-swlin skewsymmetric map*

□1(v1) □1(v2 ) $□_i(v_2$
($□_1$ □$□_2$ □$□_3$)($v_1$,$v_2$ ) = $det_{2,□}□_i(v_j)$ = $□_2(v_1)$ $□^2(v^2)$ □i □ i i □□ii12)
□ (v ) ((vv11)) 1
3 1 = $□_i(v_2$
(i1,i2 )□I23,□i i □□ □3(v2 ) i1< 2 1 )
1 2 2 2

and $□_1$ □$□_2$ □$□_3$ is a bilinear alternating map on the vectors $v_1$,$v_2$ .
Let $x^i$ :$□^n$ □□ be the function which assigns to each point of $□^n$ its $i^{th}$ -coordinate. Then $(dx^i)_p$ is a linear map in $(□^n)^□$ and the set $\{(dx^i)_p; i=1,□,n\}$ is the dual basis of the standard $\{(e_i)_p\}$. The element $(dxi^1)_p$ □□□ $(dxit)_p$ is denoted by $(dxi1$ □□□$dxit)_p$ and belongs to $□^k_t(□^n_p)^□$ .

**Theorem 4.1** *The set $\{(dxi^1$ □□□$dxit)_p\}$, $i_1$,□,$i_t$ □$I^n_t$ is a basis for $□^k_t(□^n_p)^□$ .* *Proof.* the elements of $\{(dxi^1$ □□□ $dx^{it})_p\}$ are linearly independent. In fact, suppose
i i
□ $a_{i1,□,it}$ $dx^1$ □□□$dx^t$ = 0
i1,□,it□$It^n$
then, for any ($e_j$ ,□,$e_j$ ) , with $j_1$,□, $j_k$□$I^n_k$ , it follows
1 k

□ $a_{i1,\square,it}$ $dxi^{1}$ □□□$dx^{it}$ $(e_{j1}$ ,□,$e_{jk}$ )
□,it□It n i1
i i
dx 1e j □□dx 1e j
1 k
= □ ai1,□,it □□□□□
i1,□,it□It ndxit e j1 □□dxi t e jk

i □ i
$\square_{j1}$ □ $\square_{j\,1}\,k$
1 □ □
□ i
= □ a i1,□,it □ij1 t □jt k
i ,□,i □ $I^n$
1 t t
= □□r1,□,rt
ar1,□,rt r1,□,rt □(Itn )
r ,□,r j1,□, jk
1 t
= 0

□n□
Without loss of generality, suppose $\square_{r,\square,r}$ all equal, then the $\square_{\square k}$ □□□ equations
1 t
□
n n
$\square_{r,\square,r}$ $a_{r1,\square,rt}$ = 0, $r1,\square,r_t$ $\square(I_t)_{j1,\square,jk}$, $j1,\square,j_k$ □$I_k$, are a linear omogeneous full rank system, so it has only the
1 t trivial solution. That is $a_{i,\square,i}$ = 0.
1 t
The set {(dxi ¹ □□□ dxit ) $_p$} spans $\square^{k}{}_{t}(\square^{n}{}_{p})^{\square}$, in other words any □□$\square^{k}{}_{t}(\square^{n}{}_{p})^{\square}$ may be written
□= □ $a_{i1,\square,it}$ dxi ¹ □□□$dx^{it}$ $i_1,\square,i_t$ □$I_t{}^n$
i1,□,it□It $^n$
in fact, if
□ = □ □($e_{i1}$ ,□,$e_{it}$ )$dx^{i1}$ □□□$dxi^t$
i1,□,it□It $^n$
then □($e_i$ ,□,$e_i$ ) =□($e_i$ ,□,$e_i$ ) for all $i_1,\square,i_t$ □$I_t{}^n$ , so □ =□. Setting □($e_i$ ,□,$e_i$ ) = $a_{i,\square,i}$ , it
1 t 1 t 1 t 1 t
follows the expression of □. □
i i
The above proposition generalizes the known theorem about the basis {$dx^1$ □□□$dx^k$} of the space $\square^{k}(\square^{n}{}_{p})^{\square}$ , see [1].

**Theorem 4.2** *The linear spaces* $\square^{k}{}_{t}(\square^{n}{}_{p})^{\square}$ *and* $\square^{k}(\square^{n}{}_{p})^{\square}$ *coincide.*

*Proof.* Let□= (□₁ □□□□$\square_t$)($v_1$,□,$v_k$)□□$\square^{k}{}_{t}(\square^{n}{}_{p})^{\square}$, then

□$_i$ ($v_1$) □ □$_i$ ($v_k$)
□= □1 □ 1 □ □$i$ ,□,$i$ (□1 □□□□$k$
□$i$1,□,$ik$ □ □ □ )($v$1,□,$vk$ )
$i$ ,□,$i$ □$I^n$ □$ik$ = 1 $k$
1 $k$ $k$ ($v$1) □$ik$ ($vk$ ) $i_1$,□,$i_k$□$I_k^n$

so □□□$^k$ (□$^n{}_p$ )□ . Conversely, let 0 be the null function in (□$^n{}_p$ )□, then any □□□$^k$ (□$^n{}_p$ )□ may be written
as
□ = (□$_1$ □□□□$_k$)($v_1$,□,$v_k$ ) = (□$_1$ □□□□$_k$ □0□□□0)($v_1$,□,$v_k$ ) so □□□$^k{}_t$ (□$^n{}_p$ )□.
If □□□$^k{}_t$ (□$^n{}_p$ )□ , then □ may be decomposed by elements of □$^k{}_{t□j}$ (□$^n{}_p$ )□, where $k$ □ $t$ □ $j$ □ $t$ , in fact

**Theorem 4.3** *Let*□= (□$_1$ □□□□$_t$)($v_1$,□,$v_k$ )□□$^k{}_t$ (□$^n{}_p$ )□*, then*
□ $i$ ,□,$i$
□= 1 $t$□ $j$ □(□$_i$ □□□□$_i$ )($v_1$,□,$v_k$ )

($t$ □$k$)□($t$ □$k$ □ $j$ □1) $_It$ 1 $t$□ $j$
$t$□ $j$ *Proof.*

□= □ ($it$1,□□,$k$ $it$□)1 □$^I tt$□ 1 (□$_i$1 □ □□□$it$□1 )($v$1,□,$vk$ )

=□ □ □
□ $i$ ,□,$i$
= 1 $t$□ $j$ □(□$_i$ □□□□$_i$ )($v_1$,□,$v_k$ ) ($t$ □$k$)□($t$ □$k$ □ $j$ □1) $_It$ 1 $t$□ $j$

$t$□ $j$
□ $t$ □
indeed □ is the sum of □□□ $_k$ □□□ determinants, the last right side has the same number
$t$□($t$ □ $j$ □ 2) □$t$ □ $j$□□$t$ □ $j$ □1□

($t$ □$k$)□($t$ □$k$ □ $j$ □1) □□□ $k$ □□□□□□□ $t$ □ $j$ □□□
□

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